1. **Problem:** Let X be a finite dimensional normed linear space. Show that X is a Banach space.
   Solution See Corollary 2.3.2 from the book "Functional Analysis" by S Kesavan.

2. **Problem:** Let $M = \{f \in C([0,1]) : f([0, \frac{1}{2}]) = 0\}$. Let $\Phi : C([0,1])/M \to C([0, \frac{1}{2}])$ be defined by $\Phi(\pi(f)) = f|_{[0,\frac{1}{2}]}$ where π is the qutient map. Show that Φ is an onto isometry.
   Solution To show that Φ is onto, let $g \in C([0, \frac{1}{2}])$. Define a map $f : [0,1] \to [0,1]$ by $f(x) := g(x)$ if $x \in [0, \frac{1}{2}]$ and $f(x) = g(\frac{1}{2})$ for $x \in [\frac{1}{2}, 1]$. Then, clearly $f \in C([0,1])$ and $\Phi(\pi(f)) = f|_{[0,\frac{1}{2}]} = g$.
   Let the norm on $C([0,1])/M$ is denoted by $|||\cdot|||$ and is defined by $|||f+M||| := inf\{\|f+g\| : g \in M\}$ for any $f \in C([0,1])$.
   Now, $|||\pi(f)||| = inf\{\|f + g\| : g \in M\}$. Let $\lambda := \|f|_{[0,\frac{1}{2}]}\| = sup\{|f(x)| : x \in [0, \frac{1}{2}]\}$ and $S := \{\|f + g\| : f \in C([0,1]), g \in M\}$.
   For $g \in M$ and $f \in C([0,1])$, $\|f + g\| = sup\{|f(x) + g(x)| : x \in [0,1]\} = \text{Max}\{\{sup\{|f(x) + g(x)| : x \in [0, \frac{1}{2}]\}, sup\{|f(x) + g(x)| : x \in [\frac{1}{2}, 1]\}\} = \text{Max}\{\lambda, sup\{|f(x) + g(x)| : x \in [\frac{1}{2}, 1]\}\} \geq \lambda$ i.e. λ is a lower bound of S.
   Now, let $sup\{|f(x) + g(x)| : x \in [0, \frac{1}{2}] = c_1$ and $sup\{|f(x) + g(x)| : x \in [\frac{1}{2}, 1]\} = c_2$. If $c_1 \geq c_2$ then we choose $g = 0$ and get $\inf(S) = \lambda$. If $c_1 < c_2$ then assume that f attains c_2 at some points. Let p be a point in $[\frac{1}{2}, 1]$ such that $f(p) = c_2$ and $f(x) < c_2$ for $x < p$. Now, construct a function $g \in M$ such that g is a line joining the poins $(\frac{1}{2}, 0)$ and $(0, -c_2)$ and $g(x) = -c_2$ for $x > p$. Then $\|f + g\| = \lambda$. And hence, $\lambda = \inf(S) = |||\pi(f)|||$.

3. **Problem:** For $f \in C([0,1])$ define $\|f\|_1 = \int |f|dx$. Show that this is a norm. Show that this norm is not equivalent to the supremum norm.
   Solution To show that $\|\|_1$ is a norm it is enough to show that $\|f\|_1 = 0 \implies f = 0$ because other conditions follow from the properties of integration of a continuous function on $[0,1]$. Let $\|f\|_1 = 0$. Now assume that $f \neq 0$ i.e. there exists a point $x \in [0,1]$ such that $f(x) \neq 0$. As f is continuous $|f|$ is positive in a interval around x which implies that $\|f\|_1$ is positive, a contradiction.
   For the rest part, see Example:2.3.10 from the book "Functional Analysis" by S Kesavan.

4. **Problem:** Let X and Y be Banach spaces. Let $\{T\}_n$ be a sequence of a compact operators. Suppose $T \in L(X,Y)$ and $\|T - T_n\| \to 0$. Show that T is a compact operator.
   Solution See Proposition 8.1.1 from the book "Functional Analysis" by S Kesavan.

5. **Problem:** Let $\{f\}_n \subset L^2([0,1])$ be an orthonormal sequence. Define $\Psi : L^2([0,1]) \to l^2$ by $\Psi(f) = \left(\int f f_n dx\right)_{n\geq 1}$. Show that Ψ is an onto map and Ψ^* is a one-to-one map.
   Solution $\Psi(f) = \left(\int f \bar{f}_n dx\right)_{n\geq 1} = (< f, f_n >)_{n\geq 1}$ is well defined as $\sum |\langle f, f_n\rangle|^2 \leq \|f\|^2$ using Bessel;s inequality.
   $\Psi(f_n) = e_n$ where $\{e_n\}$ is the standard basis for l^2 and hence Ψ is onto. Also, $\langle \Psi(f), e_n\rangle = \langle f, f_n\rangle$ for any $f \in L^2([0,1])$.Therefore, $\Psi^*(e_n) = f_n$ and hence Ψ^* is one-to-one because $\{f_n\}$ is a orthonormal set.

6. **Problem:** Let H be a Hilbert space. Show that $N \in L(H)$ is a normal operator if and only if $\|N(x)\| = \|N^*(x)\|$ for all $x \in H$. Hence or otherwise show that there exists a $S \in L(H)$ such that $SN = N^*$.
   Solution $N \in L(H)$ is a normal operator i.e. $NN^* = N^*N$.
   $\|N(x)\|^2 = \langle N(x), N(x)\rangle = \langle N^*N(x), x\rangle = \langle NN^*(x), x\rangle = \langle N^*(x), N^*(x)\rangle = \|N^*(x)\|^2$ for all $x \in H$.
   Now, define a map $S : N(H) \to H$ by $S(N(x)) := N^*(x)$.
   $\|S(N(x))\| = \|N^*(x)\| = \|N(x)\| \implies \|S\| \leq 1$

Therefore using Hahn Banach theorem, we a get a map denote it by S also satisfying $SN(x) = N^*(x)$ for all $x \in H$.

7. **Problem:** Let X be a Banach space. Let $P \neq Q \in L(X)$ be projections. Show that P^*, Q^* are projections in $L(X^*)$. If $PQ = QP$ show that $\|P - Q\| \geq 1$.
Solution Let X be a Banach space. $P \in L(X)$ is said to be projection if $P^2 = P$ and adjoint of $A \in L(X)$ is denoted by $A^*(\in L(X^*))$ and is defined by $A^*(y^*)(x) := y^*(Ax)$ where $x \in X$ and $y^* \in X^*$.
Suppose $A, B \in L(X)$ then $(AB)^* = B^*A^*$. Now as $P^2 = P \implies (P^*)^2 = P^*$ i.e. P^* is a projection. Similarly, Q^* is also a projection.
Now, $PQ = QP \implies P(1-Q) = (1-Q)P \implies P(1-Q)$ is a projection. Similarly, $Q(1-P)$ is a projection. As, $P \neq Q$ and $PQ = QP$, therefore $P(1-Q) \neq Q(1-P)$ and atleast, one of them is nonzero. Let $P(1-Q)$ is nonzero i.e. there is a unit vector x in its range and so, $(P-Q)x = (P-Q)P(1-Q)x = x$ which implies $\|P - Q\| \geq 1$.